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D-branes in an AdS_3 background

Sonia Stanciu

Theoretical Physics Group, Blackett Laboratory
Imperial College, London SW7 2BZ, UK
E-mail: s.stanciu@ic.ac.uk

ABSTRACT: We study the possible D-brane configurations in an $AdS_3 \times S^3 \times T^4$ background with a NS-NS B field. We use its WZW model description and the boundary state formalism, and we analyse the bosonic and the N=1 supersymmetric cases separately. We also discuss the corresponding classical open string sigma model. We determine the spacetime supersymmetry preserved by the supersymmetric D-brane configurations.

KEYWORDS: D-branes, Conformal Field Models in String Theory, Sigma Models.

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Contents

1. Introduction

In the framework of the AdS/SYM correspondence conjectured by Maldacena in [1], important particles and vacuum configurations (including interaction vertices) in the gauge theory are described, on the string theory side, by D-brane configurations (wrapped or not) in the corresponding type IIB background. Thus in [2] Witten showed how various D-brane configurations in the $AdS_5 \times S^5$ and $AdS_5 \times \mathbb{RP}^5$ backgrounds are associated with the baryon vertex and the Pfaffian particle, respectively, in the four-dimensional gauge theory at the boundary of AdS_5. Later, in [3, 4], such D-brane configurations were obtained as classical solutions of the Born-Infeld action.

Although the general belief seems to be that the AdS/SYM connection only makes sense in the Higgs branch of the SYM theory, a proposal was made [5] to identify certain vacuum configurations of the large N limit of the $\mathcal{N} = 4$ SU(N) SYM corresponding to the Coulomb branch with particular D3-brane configurations in the bulk of the $AdS_5 \times S^5$, whereby the D3-branes have worldvolumes parallel to the AdS boundary. This suggestion was further pursued in [6] where, by using a low energy analysis, more general D3-brane configurations were analysed in the AdS bulk, and the corresponding amount of supersymmetry preserved by these D-brane configurations was determined.

In the face of these facts it appears desirable to carry out a microscopic SCFT analysis of the possible D-brane configurations in various AdS-type backgrounds. Such an analysis however would rely essentially on the explicit knowledge of the CFT underlying these string backgrounds. Unfortunately, despite the progress made in constructing type IIB string theories with RR background fields on AdS spaces [7]–[12] (in the framework of the Green-Schwarz formalism), we are still lacking a satisfactory worldsheet description for most of these backgrounds. The only tractable case so far remains that of type IIB string theory on $AdS_3 \times K$, where K is a compact manifold, and we allow only a NS-NS B field. These backgrounds have received a great deal of attention recently, in the cases where K is $S^3 \times T^4$ [13, 14], or $S^3 \times S^3 \times S^1$ [15], and have been studied in detail by using perturbative methods.

Here we will initiate a study of the possible D-branes which can be formulated consistently in a superstring background defined on

$$AdS_3 \times S^3 \times T^4 \,.$$

We will restrict ourselves to the case with a purely NS-NS B field. This will allow us to use the known superconformal theory (SCFT) underlying this background, in order to apply the boundary state formalism [16]–[23].

The paper is organised as follows. In section 2 we will start with a short summary describing the bosonic background, in order to set the notation and exhibit the conformal structure. In section 3 we discuss the boundary state formalism adapted to this particular model. We consider two classes of gluing conditions: they both satisfy the basic requirement of conformal invariance, but are distinguished by the amount of the bulk symmetry they preserve. In section 4 we analyse the geometry of the resulting D-brane configurations. We consider the boundary conditions that each of the two classes of gluing conditions give rise to, and determine the types of D-branes they describe. The configurations described by gluing conditions which preserve the infinite-dimensional symmetry underlying the WZW model generically describe conjugacy classes translated by elements of the group. This result agrees in part with the similar analysis in [23]. The other type of gluing conditions gives rise to more complicated D-brane configurations which include open submanifolds of dimension equal to the dimension of the target manifold, subgroups and cosets;

although a direct comparison appears to be rather difficult, the two classes of D-brane configurations obtained here bear a certain similarity to the ones obtained in [24] by analysing the open string WZW model.

In section 5 we briefly discuss the classical boundary conditions produced by the corresponding open string sigma model in order to investigate the possibility of having D-branes whose worldvolume fills the entire target space. We find that this depends crucially on whether the corresponding group is compact or not.

In section 6 we extend our analysis to the supersymmetric case, and determine which D-brane configurations admit an $N = 1$ supersymmetric generalisation. In section 7 we discuss the fraction of spacetime supersymmetry preserved by these configurations. We end with a brief discussion and summary of results, comparing our results with known D-brane configurations in other AdS backgrounds. The paper also contains an appendix (written with J.M. Figueroa-O'Farrill) in which we investigate the conjugacy classes of $\mathrm{SL}(2,\mathbb{R})$.

2. The bosonic $AdS_3 \times S^3 \times T^4$ background

The background we are interested in appears in type IIB string theory on $\mathbb{R}^4 \times \mathbb{R}^2 \times T^4$ as the near-horizon limit of a system of p fundamental strings stretched along the $\mathbb{R}^2$ factor and k NS-fivebranes stretched along the same $\mathbb{R}^2$ and wrapped on the T^4. Its metric corresponds to an exact string background consisting of a flat space piece corresponding to T^4, a level k $\mathrm{SU}(2)$ WZW theory and a level k $\mathrm{SL}(2,\mathbb{R})$ WZW theory. We therefore start with a WZW model having as the target space a direct product of group manifolds $\mathbf{G} = \mathbf{G_1} \times \mathbf{G_2}$, where $\mathbf{G_1} = \mathrm{SL}(2,\mathbb{R})$, $\mathbf{G_2} = \mathrm{SU}(2)$. The corresponding WZW action will therefore be a sum of two independent terms

$$I = I_{\mathrm{SL}(2,\mathbb{R})}[g_1] + I_{\mathrm{SU}(2)}[g_2]\,, \tag{2.1}$$

where each term is of the form

$$\frac{2}{k} I[g_i] = \int_\Sigma \langle g_i^{-1}\partial g_i, g_i^{-1}\bar{\partial} g_i\rangle + \frac{1}{6}\int_B \langle g_i^{-1}dg_i, [g_i^{-1}dg_i, g_i^{-1}dg_i]\rangle\,.$$

The full action of the theory includes, of course, a term corresponding to T^4 and describing four commuting (compact) bosons. Each of the fields g_i is a map from a closed orientable Riemann surface Σ to the Lie group $\mathbf{G}_i$, $i = 1, 2$. We denote by $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ the corresponding Lie algebra, where $\mathfrak{g}_1 = \mathfrak{sl}(2,\mathbb{R})$ and $\mathfrak{g}_2 = \mathfrak{su}(2)$. For these algebras we choose the following bases of generators: $\{X_a\}$ for $\mathfrak{g}_1$, and $\{Y_a\}$ for $\mathfrak{g}_2$ satisfying

$$[X_1, X_2] = X_3\,, \qquad [X_2, X_3] = -X_1\,, \qquad [X_3, X_1] = -X_2\,,$$

and

$$[Y_1, Y_2] = Y_3\,, \qquad [Y_2, Y_3] = Y_1\,, \qquad [Y_3, Y_1] = Y_2\,.$$

We also need to specify an invariant metric on $\mathfrak{g}$, which has a diagonal form

$$\eta = \begin{pmatrix} \eta_1 & 0 \\ 0 & \eta_2 \end{pmatrix},$$

with components $(\eta_1)_{ab} \equiv \langle X_a, X_b \rangle = \mathrm{diag}(+,+,-)$ and $(\eta_2)_{ab} \equiv \langle Y_a, Y_b \rangle = \mathrm{diag}(+,+,+)$.

We will use the following parametrisation[1] for the group manifold $\mathbf{G}$:

$$g_1 = e^{\theta_2 X_2} e^{\theta_1 X_1} e^{\theta_3 X_3}, \qquad g_2 = e^{\phi_2 Y_2} e^{\phi_1 Y_1} e^{\phi_3 Y_3}, \tag{2.2}$$

where θ_μ and ϕ_μ, $\mu = 1,2,3$, play the rôle of the spacetime fields. In terms of them (2.1) becomes a sigma-model action, with the spacetime metric and 2-form given by

$$\begin{aligned} ds_1^2 &= d\theta_1 d\theta_1 + d\theta_2 d\theta_2 - d\theta_3 d\theta_3 + 2\sinh\theta_1 d\theta_2 d\theta_3 \,, \\ dB_1 &= -\cosh\theta_1 d\theta_1 d\theta_2 d\theta_3 \,, \end{aligned}$$

on $\mathrm{SL}(2,\mathbb{R})$, and

$$\begin{aligned} ds_2^2 &= d\phi_1 d\phi_1 + d\phi_2 d\phi_2 + d\phi_3 d\phi_3 - 2\sin\phi_1 d\phi_2 d\phi_3 \,, \\ dB_2 &= \cos\phi_1 d\phi_1 d\phi_2 d\phi_3 \,, \end{aligned}$$

for $\mathrm{SU}(2)$. (The torus will have of course flat metric and no B field.)

The exact conformal invariance of this model is based, as is well known, on its infinite-dimensional symmetry group $\mathbf{G}(z) \times \mathbf{G}(\bar{z})$ characterised by the conserved currents $I(z) = -k\partial g_1 g_1^{-1}$ and $\bar{I}(\bar{z}) = k g_1^{-1}\bar{\partial} g_1$ corresponding to $\mathfrak{sl}(2,\mathbb{R})$, and $J(z) = -k\partial g_2 g_2^{-1}$ and $\bar{J}(\bar{z}) = k g_2^{-1}\bar{\partial} g_2$ corresponding to $\mathfrak{su}(2)$. These currents generate an affine Lie algebra $\widehat{\mathfrak{g}}_1 \oplus \widehat{\mathfrak{g}}_2$, with $\widehat{\mathfrak{g}}_1$ described by

$$I_a(z) I_b(w) = \frac{k(\eta_1)_{ab}}{(z-w)^2} + \frac{f_{ab}{}^c I_c(w)}{z-w} + \mathrm{reg}\,, \tag{2.3}$$

where the parameter k is related to the level x of the affine algebra by $k = x + g^*$, where g^* is the dual Coxeter number. Similarly, for $\widehat{\mathfrak{g}}_2$ we have

$$J_a(z) J_b(w) = \frac{k(\eta_2)_{ab}}{(z-w)^2} + \frac{f_{ab}{}^c J_c(w)}{z-w} + \mathrm{reg}\,, \tag{2.4}$$

whereas the free bosons on the torus, described by the fields φ_i, $i = 1,2,3,4$, satisfy the standard OPEs

$$\partial\varphi_i(z)\partial\varphi_j(w) = \frac{\delta_{ij}}{(z-w)^2} + \mathrm{reg}\,, \tag{2.5}$$

[1] This parametrisation of AdS_3 is different from the one given by the Gauss decomposition,

$$g_1 = e^{\bar{\gamma}(X_2 - X_3)} e^{2rX_1} e^{\gamma(X_2+X_3)},$$

and which yields the familiar metric $ds_1^2 = \frac{1}{u^2}du^2 + u^2 d\gamma d\bar{\gamma}$, where $u = e^r$.

with similar OPEs for the antiholomorphic sector. The corresponding CFT is then described by the energy-momentum tensor

$$\mathsf{T} = \Omega_1^{ab}(I_a I_b) + \Omega_2^{ab}(J_a J_b) + \sum_{i=1}^{4}(\partial\varphi_i \partial\varphi_i)\,,$$

where Ω_1^{ab} and Ω_2^{ab} are components of the inverse of the following invariant metric

$$\Omega = \begin{pmatrix} \Omega_1 & 0 \\ 0 & \Omega_2 \end{pmatrix},$$

with the components given by $\Omega_1 = 2(k+1)\eta_1$ and $\Omega_2 = 2(k-1)\eta_2$. The central charge of this CFT is given by

$$c = \frac{3k}{k+1} + \frac{3k}{k-1} + 4\,.$$

The choice of equal levels for $\widehat{\mathfrak{g}}_1$ and $\widehat{\mathfrak{g}}_2$ was motivated by the fact that, in this case, the corresponding $N=1$ supersymmetric background is a critical superstring theory, as we will see in section 6.

3. Boundary states

The boundary state formalism (see, e.g., [25, 26, 27, 28]) has become in the last years one of the main approaches to the study of D-branes in type II string backgrounds [16]–[21]. Using an explicit knowledge of the CFT underlying a given string background, one describes a D-brane configuration through a set of boundary conditions, relating the left- and the right- moving conformal structures in such a way that conformal invariance is preserved. In the case of a background described by a WZW model the fields in terms of which the conformal structure of the model is realised are the affine currents. It is therefore convenient to impose the boundary conditions on these currents in order to have under control the conformal invariance of the resulting configurations.

In what follows we will consider two different classes of gluing[2] conditions which give rise to different configurations of D-branes. Both of them will be defined in terms of a Lie algebra automorphism, $R:\mathfrak{g}\to\mathfrak{g}$, which preserves the metric η; in other words we have

$$[R(Z_a), R(Z_b)] = R([Z_a, Z_b])\,, \tag{3.1}$$

$$R^T \eta R = \eta\,, \tag{3.2}$$

[2]In order to avoid confusion between the boundary conditions satisfied by the chiral currents, J and $\bar{J}$, and the ones satisfied the field g or its components, we will refer to the boundary conditions on the chiral currents as 'gluing conditions', reserving the term of boundary conditions to the fields themselves.

where $\{Z_a\}$ is a given basis in $\mathfrak{g}$, in terms of which R is given by $R(Z_a) = Z_b R^b{}_a$. We can now define the two classes of gluing conditions[3] as being the following:

(i) *type-N gluing conditions*, which can be thought of as a generalisation of the Neumann conditions, are given by

$$J_a(z) + R^b{}_a \bar{J}_b(\bar{z}) = 0\,, \tag{3.3}$$

at the boundary of the worldsheet. Notice that these gluing conditions do not preserve the infinite-dimensional affine symmetry of the current algebra (correcting a statemnet made in [22]);

(ii) *type-D gluing conditions*, which can be thought of as a generalisation of Dirichlet conditions, read

$$J_a(z) - R^b{}_a \bar{J}_b(\bar{z}) = 0\,. \tag{3.4}$$

By contrast, this type of gluing conditions does preserve the current algebra of the bulk theory.

These gluing conditions have to satisfy the basic consistency requirement, which is conformal invariance. This means that the holomorphic and the antiholomorphic sectors are related by an automorphism of the corresponding CFT. In the bosonic case, since the automorphism group of the Virasoro algebra is trivial, we impose

$$\mathsf{T}(z) = \bar{\mathsf{T}}(\bar{z})\,,$$

at the boundary. In this case, the requirement of conformal invariance translates into the condition

$$R^T \Omega R = \Omega\,, \tag{3.5}$$

for either of the two types of gluing conditions.

Ignoring the flat part of the target space — that is, T^4 — for which the possible D-branes are known, we take $\mathfrak{g}$ to be the direct sum of the two Lie algebras $\mathfrak{g}_1 = \mathfrak{sl}(2,\mathbb{R})$ and $\mathfrak{g}_2 = \mathfrak{su}(2)$. Because these Lie algebras are different real forms of the same complex Lie algebra $\mathfrak{sl}(2,\mathbb{C})$, there is no nontrivial homomorphism between them. This implies that the matrix of gluing conditions R, defined by the automorphism $R : \mathfrak{g}_1 \oplus \mathfrak{g}_2 \to \mathfrak{g}_1 \oplus \mathfrak{g}_2$, must take a block-diagonal form

$$R = \begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix}, \tag{3.6}$$

where $R_1 : \mathfrak{g}_1 \to \mathfrak{g}_1$ and $R_2 : \mathfrak{g}_2 \to \mathfrak{g}_2$. Then from (3.5) and (3.2) we deduce that R_1 and R_2 must separately preserve the metric on $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SU}(2)$ respectively, and therefore R_1 defines an element of $\mathrm{O}(2,1)$, whereas R_2 is an element of $\mathrm{O}(3)$.

[3] We will be working here in the open string picture.

On the other hand, from (3.1) it follows that both R_1 and R_2 are Lie algebra automorphisms, corresponding to $\mathfrak{sl}(2,\mathbb{R})$ and $\mathfrak{su}(2)$, respectively. Explicitly, each of these two automorphism conditions translates into a condition on the corresponding matrix, that is

$$\det(R_1) = \det(R_2) = 1\,,$$

which makes R_1 belong to $\mathrm{SO}(2,1)$, and R_2 to $\mathrm{SO}(3)$.

These results can be summarised as follows. We have identified two classes of gluing conditions on the group manifold $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SU}(2)$ which preserve conformal invariance. Each of them is described in terms of an automorphism of the Lie algebra $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{su}(2)$, but only the type-D gluing conditions (3.4) preserve also the infinite-dimensional symmetry of the current algebra of the bulk theory. In what follows we will analyse in detail both these types of configurations in order to identify the D-branes they describe.

4. D-brane solutions and geometry

The geometric interpretation of the gluing conditions defined on the chiral currents of the WZW theory in terms of D-brane configurations is arguably the most subtle issue of this approach. The precise statement of the problem is the following: given a set of gluing conditions for the chiral currents and a fixed but otherwise arbitrary point g in the target group manifold, find the possible D-branes which pass through g and are described by these gluing conditions.

The difficulty lies with the fact that the gluing conditions imposed on the affine currents, despite being a natural nonabelian generalisation of the boundary conditions in a free theory are *not*, strictly speaking, boundary conditions. The flat space boundary conditions are defined in the tangent space of the target manifold and therefore the eigenvalues and eigenvectors of the corresponding matrix R identify the Neumann and Dirichlet directions. In the group manifold case the gluing conditions take values in the tangent space of $\mathbf{G}$ *at the identity*, that is $T_e\mathbf{G} \equiv \mathfrak{g}$, because the currents themselves are Lie algebra valued objects. Hence, in order to interpret geometrically the algebraic gluing conditions we must first of all 'translate' them into boundary conditions in $T_g\mathbf{G}$, and then determine what the Neumann and Dirichlet directions are in this case.

In case of the $AdS_3 \times S^3$ background, the matrix R has a block diagonal form (3.6), which allows us to analyse the D-brane configurations on AdS_3 and S^3 separately. Thus R_1, which is an element of $\mathrm{SO}(2,1)$ can be thought of as a Lorentz transformation in $\mathbb{R}^{2,1}$. In three dimensions, any Lorentz transformation leaves a vector invariant. Depending on the causal type of this vector, we can distinguish three types of Lorentz transformations:

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(i) *spatial rotations*, typically of the form

$$R_1 = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \alpha \in \mathbb{R}. \tag{4.1}$$

In this case R_1 has generically one $+1$ eigenvalue, with the corresponding eigenvector being time-like;

(ii) *boosts*, typically of the form

$$R_1 = \begin{pmatrix} \cosh\alpha & 0 & \sinh\alpha \\ 0 & 1 & 0 \\ \sinh\alpha & 0 & \cosh\alpha \end{pmatrix}, \qquad \alpha \in \mathbb{R}. \tag{4.2}$$

In this case R_1 has generically one $+1$ eigenvalue, but with the corresponding eigenvector being space-like;

(iii) *null rotations*, typically of the form

$$R_1 = \begin{pmatrix} 1 & -a & a \\ a & 1-\frac{1}{2}a^2 & \frac{1}{2}a^2 \\ a & -\frac{1}{2}a^2 & 1+\frac{1}{2}a^2 \end{pmatrix}, \qquad a \in \mathbb{R}. \tag{4.3}$$

In this case R_1 has generically a $+1$ eigenvalue whose eigenvector is light-like. In contrast with the other two cases, a null rotation is not diagonalisable over the complex numbers. This makes the geometric interpretation of the corresponding configuration relatively difficult.

By contrast, a D-brane configuration in SU(2) is described by an element R_2 in SO(3) that is, a rotation in $\mathbb{R}^3$, typically of the form

$$R_2 = \begin{pmatrix} \cos\beta & \sin\beta & 0 \\ -\sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \beta \in \mathbb{R}. \tag{4.4}$$

It is generically characterised by a $+1$ eigenvalue, whose eigenvector corresponds to the direction which is left invariant by the rotation.

Type-N configurations We now arrive at the basic fact on which our geometric interpretation of the gluing conditions is based. Here we will only state this result (details will appear elsewhere [29]) and proceed to apply it to the case of the group manifold corresponding to $AdS_3 \times S^3$.

Let us parametrise the group manifold $\mathbf{G}$ by introducing the coordinates X^μ, with $\mu = 1, \ldots, \dim \mathbf{G}$; we also introduce the left- and right-invariant vielbeins, e and $\bar{e}$, defined by

$$g^{-1}dg = e^a{}_\mu \, dX^\mu Z_a \,, \qquad dgg^{-1} = \bar{e}^a{}_\mu \, dX^\mu Z_a \,.$$

The gluing conditions (3.3) will then give rise to the following boundary conditions at a generic point g in $\mathbf{G}$:

$$\partial X^\mu = \tilde{R}(g)^\mu{}_\nu \bar{\partial} X^\nu \,,$$

where the matrix of boundary conditions $\tilde{R}(g)$ is given by

$$\tilde{R}(g) = \bar{e}^{-1} R e \,.$$

Notice that $\tilde{R}(g)$, which describes the boundary conditions at a given point in the target space, depends on that point through the invariant vielbeins. One can now identify the Neumann and Dirichlet directions by analogy with the flat space case. At a given point, a Dirichlet boundary condition corresponds to a -1 eigenvalue of the matrix $\tilde{R}(g)$, which means that the directions normal to the worldvolume of the D-brane are spanned by the corresponding eigenvectors of $\tilde{R}(g)$. All the other eigenvalues describe Neumann boundary conditions (in the presence of a B field) and the corresponding eigenvectors span the tangent space of the worldvolume of the D-brane.

We now start with $\mathrm{SL}(2,\mathbb{R})$, and consider the simplest possible case where $R_1 = \mathbb{1}$. In this case the boundary conditions at a point $g_1(\theta_\mu)$ will read

$$\partial \theta^\mu = \tilde{R}_1(g_1)^\mu{}_\nu \bar{\partial} \theta^\nu \,,$$

where $\tilde{R}_1(g_1) = \bar{e}_1^{-1} e_1$. If we evaluate the matrix of boundary conditions at the identity we obtain $\tilde{R}_1(e) = \mathbb{1}$, which indicates that we have three Neumann directions spanning the whole $\mathfrak{sl}(2,\mathbb{R})$. In other words, the identity in $\mathrm{SL}(2,\mathbb{R})$ belongs to a D2-brane.

If we now move away from the identity, $\tilde{R}_1$ will no longer be $\mathbb{1}$. Instead, we obtain that $\tilde{R}_1$ always has one $+1$ eigenvalue and two complex conjugate eigenvalues. Hence, at generic points in the group manifold, $\tilde{R}_1(g_1)$ will still give rise to three Neumann directions; however there will be a submanifold of $\mathrm{SL}(2,\mathbb{R})$ where $\tilde{R}_1(g_1)$ will have at least one -1 eigenvalue (two, in fact, since $\det \tilde{R}_1 = 1$). In other words, at each point on this particular submanifold we will have one Neumann and two Dirichlet directions. This submanifold can be described as the zero locus $\mathcal{F}_1$ of a function, $F_1(g_1) \equiv \mathrm{Tr}\, \tilde{R}_1(g_1) + 1$, which in our parametrisation is given by

$$\begin{aligned} F_1 = {} & 1 + \cosh\theta_1 \cosh\theta_2 + \cosh\theta_2 \cos\theta_3 + \cos\theta_3 \cosh\theta_1 + \\ & + \sinh\theta_1 \sinh\theta_2 \sin\theta_3 \,. \end{aligned}$$

One can show that F_1 is a class function; that is, $F_1(h^{-1}gh) = F_1(g)$ for all group elements g, h. Indeed, if we consider the vector fields that generate the adjoint action of the group $\mathrm{SL}(2,\mathbb{R})$

$$H_a(g_1) = \left((\bar{e}_1^{-1})^\mu{}_a - (e_1^{-1})^\mu{}_a\right) \partial_\mu \,, \qquad a = 1,2,3, \tag{4.5}$$

we can check that F_1 is annihilated by them, that is

$$H_a(g_1) \cdot F_1(g_1) = 0\,,$$

for all $a = 1, 2, 3$. Hence we have that $\mathcal{F}_1$ consists of adjoint orbits — that is, conjugacy classes.

At every point in $\mathcal{F}_1$ we have one Neumann and two Dirichlet boundary conditions. The vector field corresponding to the Neumann direction therefore spans the worldline of a D-particle. In order for this picture to be consistent we must verify that the worldlines of these D0-branes lie within $\mathcal{F}_1$. Indeed, a straightforward calculation shows that we have

$$V_1(g_1) \cdot F_1(g_1)|_{F_1(g_1)=0} = 0\,,$$

which allows us to conclude that V_1 is tangent to $\mathcal{F}_1$.

What happens now if we start with a matrix R_1 which is different from the identity? First of all we recall that $\mathfrak{sl}(2,\mathbb{R})$ is a simple Lie algebra, and therefore R_1 is an inner automorphism; hence it can be identified with Ad_{r_1}, for some group element r_1. As we will see in the next paragraph, the effect of an inner automorphism at the level of the gluing conditions is a translation in the group manifold. More precisely, the submanifold on which $\tilde{R}_1 = \bar{e}_1^{-1} R_1 e_1^{-1}$ has -1 eigenvalues, which we denote by $\mathcal{F}_1^{(r_1)}$, is the zero locus of a function $F_1^{(r_1)}$ which satisfies $F^{(r_1)}(g) = F(g r_1^{-1})$. Thus $\mathcal{F}_1^{(r_1)}$ is nothing but the translation of the previous $\mathcal{F}_1$ by the group element r_1, that is $\mathcal{F}_1^{(r_1)} = \mathcal{F}_1 r_1$. Hence through every point in $\mathcal{F}_1 r_1$ passes a D0-brane whose worldline lies on $\mathcal{F}_1 r_1$. Moreover, the Neumann eigenvectors tangent to the worldline of the D0-branes in $\mathcal{F}_1 r_1$ can be obtained by translating accordingly the corresponding Neumann eigenvectors tangent to the D0-branes in $\mathcal{F}_1$.

A particularly interesting case is the one where R_1 itself has two -1 eigenvalues (this can be obtained by taking R_1 of the form (4.1) with $\alpha = \pi$). In this case, the corresponding surface $\mathcal{F}_1 r_1$ passes through the identity element in $\mathrm{SL}(2,\mathbb{R})$, and therefore there exists a particular D0-brane whose worldline passes through the identity. Its tangent vector takes a particularly simple form at the identity, being given by $V_1 = \partial_{\theta_3}$, as expected. The worldline of this D0-brane is nothing but the subgroup of $\mathrm{SL}(2,\mathbb{R})$ generated by X_3. Moreover, the translation of this particular solution gives rise to D0-brane configurations whose worldlines are cosets in $\mathrm{SL}(2,\mathbb{R})$.

We can now analyse the SU(2) case in a similar fashion. If we start with $R_2 = \mathbb{1}$, the boundary conditions at a point $g_2(\phi_i)$ will read

$$\partial\phi^\mu = \tilde{R}_2(g_2)^\mu{}_\nu \bar{\partial}\phi^\nu\,,$$

where $\tilde{R}_2(g_2) = \bar{e}_2^{-1} e_2$. At the identity we have $\tilde{R}_2(e) = \mathbb{1}$, which gives three Neumann directions spanning $\mathfrak{su}(2)$. We can therefore conclude that the identity in SU(2) belongs to an euclidean D2-brane. Away from the identity, $\tilde{R}_2$ will have one

+1 and two complex conjugate eigenvalues, which generically describe three Neumann directions. Similarly to the previous case, there will be a submanifold $\mathcal{F}_2$ of SU(2) where $\tilde{R}_2$ has two -1 eigenvalues. This submanifold can be described as the zero locus of the function $F_2(g_2) \equiv \mathrm{Tr}\,\tilde{R}_2(g_2) + 1$ which reads

$$\begin{aligned} F_2 = 1 &+ \cos\phi_1 \cos\phi_2 + \cos\phi_2 \cos\phi_3 + \cos\phi_3 \cos\phi_1 + \\ &+ \sin\phi_1 \sin\phi_2 \sin\phi_3 \,. \end{aligned}$$

As before, one can show that F_2 is a class function, and hence that $\mathcal{F}_2$ consists of conjugacy classes.

One can show, along the same lines, that $\mathcal{F}_2$ is foliated by the worldlines of the euclidean D0-branes whose tangent vectors are given by the eigenvectors V_2 of $\tilde{R}_2$ corresponding to the +1 eigenvalue.

Finally, if instead of $R_2 = \mathbb{1}$ we take $R_2 = \mathrm{Ad}_{r_2}$ (since, as before, R_2 is an inner automorphism) the resulting D-brane configurations can be understood as translations in the group manifold of the ones obtained in the case $R_2 = \mathbb{1}$. Thus, we obtain in particular euclidean D0-branes whose worldlines lie in $\mathcal{F}_2 r_2$.

Type-D configurations This type of gluing gluing conditions (3.4) have been recently analysed in [23], where the resulting D-brane configurations have been identified with conjugacy classes. Here however, by using a slightly different point of view, we will arrive at partially different conclusions.

As we mentioned before, the reason for which the gluing conditions (3.4) cannot immediately be interpreted as boundary conditions in the target space is the fact that the currents themselves are Lie algebra valued objects. In order to obtain a boundary condition from the gluing condition (3.4), we must translate the currents, which take values in $\mathfrak{g}$, into objects taking values in $T_g\mathbf{G}$. In this way one obtains [29]

$$\partial g = \mathbf{R}(g)\bar{\partial} g\,, \tag{4.6}$$

with the matrix of boundary conditions given by

$$\mathbf{R}(g) = -(\rho_g)_* \circ R \circ (\lambda_g)_*^{-1}\,,$$

where λ_g and ρ_g stand for left- and right-multiplication by g in $\mathbf{G}$. The condition (4.6) now takes place in $T_g\mathbf{G}$, and it is the corresponding, point-dependent matrix $\mathbf{R}(g)$ which determines the Neumann and Dirichlet directions. Indeed, at a given point g in $\mathbf{G}$, a Dirichlet boundary condition corresponds to a -1 eigenvalue of $\mathbf{R}(g)$, which means that the directions normal to the worldvolume of the D-brane are spanned by the corresponding eigenvectors of $\mathbf{R}(g)$. All the other eigenvalues describe Neumann boundary conditions and the corresponding eigenvectors span the tangent space of the worldvolume of the D-brane.

If R is taken to be the identity matrix, then $\mathbf{R}(g) = -\operatorname{Ad}_{g^{-1}}$, and the corresponding D-branes can be identified with the conjugacy classes of the group $\mathbf{G}$ [23]. Indeed, in this case, and provided that the metric G restricts nondegenerately to the conjugacy class C of g, the tangent space at g splits into the tangent space to the conjugacy class and its perpendicular complement, which can be identified with the tangent space to the centraliser subgroup Z of g:

$$T_g\mathbf{G} = T_gC \oplus T_gZ \quad \text{with} \quad T_gC \perp T_gZ\,.$$

Moreover $\operatorname{Ad}_{g^{-1}}$ restricts to the identity on T_gZ, which means that the Dirichlet directions span T_gZ. Furthermore, the Neumann directions span T_gC, and hence the worldvolume of the D-brane can be identified with C.

Let us now consider the case of an arbitrary R. In our case, since both $\mathfrak{sl}(2,\mathbb{R})$ and $\mathfrak{su}(2)$ are simple Lie algebras, we can restrict ourselves to the case where R is an inner automorphism, and hence can be identified with Ad_r, for some group element r. Therefore the corresponding boundary conditions can be written in the following form

$$\partial\tilde{g} = -\operatorname*{Ad}_{\tilde{g}^{-1}}\bar{\partial}\tilde{g}\,,$$

with $\tilde{g} = gr^{-1}$. This implies that the corresponding D-brane lies along the right-translate Cr of the conjugacy class of g by the element r. This result contradicts the statement made in [23] according to which inner automorphisms, being a "symmetry of the model" cannot result in D-brane configurations different from the the ones already described by $R = \mathbb{1}$. Although inner automorphisms are symmetries of the background, they are not necessarily symmetries the theory containing a D-brane. This fact is not surprising, as D-branes break some of the bulk symmetries even in flat space (e.g., translational symmetry).

We can now specialise to our particular background where, as usual, we will consider the two groups separately. The conjugacy classes of $\mathrm{SU}(2)$ are well known, and have been recently discussed in [23]. We have listed them in table 1. They are parametrised by $S^1/\mathbb{Z}_2$, which we can understand as the interval $\theta \in [0,\pi]$. The conjugacy classes corresponding to $\theta = 0,\pi$ are points, corresponding to the elements $\pm e$ in the centre of $\mathrm{SU}(2)$, whereas the classes corresponding to $\theta \in (0,\pi)$ are spheres. If we picture $\mathrm{SU}(2)$, which is homeomorphic to the 3-sphere, as the one-point compactification of $\mathbb{R}^3$ where the sphere at infinity is collapsed to a point, the foliation of $\mathrm{SU}(2)$ by its conjugacy classes coincides with the standard foliation of $\mathbb{R}^3$ by 2-spheres with two degenerate spheres at the origin and at infinity.

Class	Element	Topology
$\mathcal{C}'_e$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	point
$\mathcal{C}'_{-e}$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	point
$\mathcal{C}'_\theta$	$\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$	S^2

Table 1: $\mathrm{SU}(2)$ conjugacy classes.

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Let us now turn to $\mathrm{SL}(2,\mathbb{R})$. Its conjugacy classes are computed in the appendix and can be labelled by eight types of 2×2 matrices. Those classes which are metrically nondegenerate can be interpreted as D-branes and are listed in table 2. As in the case of $\mathrm{SU}(2)$ we have two point-like D-branes corresponding to the two elements in the centre of $\mathrm{SL}(2,\mathbb{R})$ as well as a family of euclidean D-strings with planar topology, but in addition there is also a family of D-strings with cylindrical topology.

Class	Element	Topology
$\mathcal{C}_e$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	point
$\mathcal{C}_{-e}$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	point
$\mathcal{C}_\theta$	$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$	$\mathbb{R}^2$
$\mathcal{C}_\lambda$	$\begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix}$	$\mathbb{R}\times S^1$

Table 2: $\mathrm{SL}(2,\mathbb{R})$ D-branes based on conjugacy classes.

Let us now summarise our findings so far. We have seen that type-N gluing conditions give rise to D5-, D3-, and D1-branes in $AdS_3\times S^3$ whose worldvolumes are of the form $N_1\times N_2$, with N_1 and N_2 three- or one-dimensional submanifolds of AdS_3 and S^3, respectively. Moreover, particular solutions for N_1 and N_2 include subgroups and cosets of $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SU}(2)$, respectively. By contrast, type-D gluing conditions, for a given $R=\mathrm{Ad}_r$, describe D-branes whose "worldvolumes" are shifted conjugacy classes of the form $\mathcal{C}r_1\times\mathcal{C}'r_2$, which can be 0-, 2- or 4-dimensional.

5. Relation to the sigma model approach

One of the most surprising results of the boundary state analysis of the possible D-branes in $AdS_3\times S^3$ is the absence of D-brane configurations which fill the entire group manifold. In this section we pause for a moment our analysis via the boundary state approach to consider the classical sigma model which corresponds to our WZW theory. Our main aim here is to investigate the possibility of having D-branes which fill the whole target space, that is $AdS_3\times S^3$.

The action of a generic WZW model on a 2-space with a disc topology with an additional interaction (1-form field A) at the boundary reads

$$S=\int_\Sigma\langle g^{-1}\partial g,g^{-1}\bar\partial g\rangle+\int_\Sigma g^*B+\int_{\partial\Sigma}g^*A\,. \tag{5.1}$$

Here the worldsheet Σ is a two-dimensional manifold with boundary $\partial\Sigma$, and B represents a particular choice for the antisymmetric tensor field. A D-brane configuration is characterised in this setting (for more details see [24]) by a two-form α living on the worldvolume D of the D-brane (in which the boundary of the string worldsheet $\partial\Sigma$ is included), and satisfying $d\alpha=dB|_D$. Since $d(B-\alpha)|_D=0$, one can define locally the one-form potential A such that $dA=B-\alpha$. S may be viewed as a special case of an action for an open string propagating on a group manifold and coupled to A at the boundary. In the case of $AdS_3\times S^3$ the action consists of

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two independent terms, $S_1 + S_2$, such that each of them is of the form (5.1), only with different target spaces, corresponding to the two groups, $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SU}(2)$, respectively. However for the most part we will work with the generic form of the action (5.1), and we will consider the two components separately only at the very end.

The infinitesimal variation of S contains a bulk term (yielding the same equations of motion as in the closed string case) and a boundary term which reads

$$\int_{\partial\Sigma} d\tau (g^{-1}\delta g)^a \left[\eta_{ab}(g^{-1}\partial_\sigma g)^b - i\alpha_{ab}(g^{-1}\partial_\tau g)^b\right]\Big|_{\sigma=0}^{\sigma=\pi} = \int_{\partial\Sigma} d\tau \delta X^\mu p_\mu \Big|_{\sigma=0}^{\sigma=\pi} ,$$

where η is the generic metric on the group manifold and X^μ are the coordinates introduced in the previous section. We have denoted by p_μ the component of the 2-momentum normal to the boundary $\partial\Sigma$ which is given by

$$p_\mu = G_{\mu\nu}\partial_\sigma x^\nu - i\alpha_{\mu\nu}\partial_\tau x^\nu ,$$

where $G_{\mu\nu} = e^a{}_\mu \eta_{ab} e^b{}_\nu$, $\alpha_{\mu\nu} = e^a{}_\mu \alpha_{ab} e^b{}_\nu$.

Having Neumann boundary conditions in all directions amounts to imposing $p_\mu|_{\partial\Sigma} = 0$, for all μ. In order to compare these conditions with the boundary conditions (3.3), we must express them in terms of the same quantities — that is, in terms of the conserved currents. For this we use the expressions of the currents in terms of the spacetime fields

$$J_a = -\eta_{ab}\bar{e}^b{}_\mu \partial X^\mu , \qquad \bar{J}_a = \eta_{ab} e^b{}_\mu \bar{\partial} X^\mu , \tag{5.2}$$

in order to rewrite p_μ as follows:

$$p_\mu = -[\delta_\mu{}^\rho - \alpha_{\mu\nu}G^{\nu\rho}]\bar{e}^a{}_\rho J_a - [\delta_\mu{}^\rho + \alpha_{\mu\nu}G^{\nu\rho}] e^a{}_\rho \bar{J}_a .$$

Then the Neumann boundary conditions take the following form

$$J + M\bar{J} = 0 , \tag{5.3}$$

where the matrix M depends on the background fields, being given by

$$M \equiv \bar{e}^{-T} \frac{\mathbb{1} + \alpha G^{-1}}{\mathbb{1} - \alpha G^{-1}} e^T . \tag{5.4}$$

Since we imposed Neumann conditions in all directions, it is natural to expect that we obtain a D-brane whose worldvolume has the same dimension as the dimension of the target space. Notice however that from this we cannot immediately deduce that the D-brane worldvolume literally fills the whole target space. We will return to this point at the end of this section.

In our case, since all the relevant quantities (that is, the background fields and the corresponding vielbeins) take a block-diagonal form with respect to the two group components, the matrix M will do so as well,

$$M = \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix}.$$

Therefore we can compute the two components separately.

5.1 The $\mathrm{SL}(2,\mathbb{R})$ component

We can now compute the matrix of boundary conditions M_1 corresponding to the open WZW action S_1 with target space $\mathrm{SL}(2,\mathbb{R})$. In order to do this we use the parametrisation of $\mathrm{SL}(2,\mathbb{R})$ given by the first expression in (2.2). Then the invariant vielbeins e_1 and $\bar{e}_2$ are given by

$$e_1 = \begin{pmatrix} \cos\theta_3 & -\cosh\theta_1 \sin\theta_3 & 0 \\ \sin\theta_3 & \cosh\theta_1 \cos\theta_3 & 0 \\ 0 & -\sinh\theta_1 & 1 \end{pmatrix}$$

$$\bar{e}_1 = \begin{pmatrix} \cosh\theta_2 & 0 & -\cosh\theta_1 \sinh\theta_2 \\ 0 & 1 & \sinh\theta_1 \\ -\sinh\theta_2 & 0 & \cosh\theta_1 \cosh\theta_2 \end{pmatrix}.$$

The corresponding background metric will be $(G_1)_{\mu\nu} = e^a{}_\mu (\eta_1)_{ab} e^b{}_\nu$. If we choose $\alpha_1 = \sinh\theta_1 d\theta_2 d\theta_3$ we find that the matrix of the boundary conditions is given by

$$M_1 = \begin{pmatrix} \cosh\theta_2 & 0 & \sinh\theta_2 \\ 0 & 1 & 0 \\ \sinh\theta_2 & 0 & \cosh\theta_2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh\theta_1 & \sinh\theta_1 \\ 0 & \sinh\theta_1 & \cosh\theta_1 \end{pmatrix} \begin{pmatrix} \cos\theta_3 & \sin\theta_3 & 0 \\ -\sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Alternatively, one can write M_1 in a more succinct form by using the adjoint action of the group, in terms of which we have $M_1 = \mathrm{Ad}(e^{\theta_2 X_2} e^{-\theta_1 X_1} e^{\theta_3 X_3})$. One can therefore see that M_1 is indeed an element of $\mathrm{SO}(2,1)$, as we obtained in section 3, where the parameters are given by the fields themselves.

5.2 The $\mathrm{SU}(2)$ component

We now turn to S_2, whose matrix of boundary conditions we denote by M_2. We use the parametrisation of $\mathrm{SU}(2)$ given by the second expression in (2.2), and compute the corresponding invariant vielbeins e_2 and $\bar{e}_2$ which read

$$e = \begin{pmatrix} \cos\phi_3 & \cos\phi_1 \sin\phi_3 & 0 \\ -\sin\phi_3 & \cos\phi_1 \cos\phi_3 & 0 \\ 0 & -\sin\phi_1 & 1 \end{pmatrix}$$

$$\bar{e} = \begin{pmatrix} \cos\phi_2 & 0 & \cos\phi_1 \sin\phi_2 \\ 0 & 1 & -\sin\phi_1 \\ -\sin\phi_2 & 0 & \cos\phi_1 \cos\phi_2 \end{pmatrix}.$$

From this we obtain the corresponding background metric, $(G_2)_{\mu\nu} = e^a{}_\mu (\eta_2)_{ab} e^b{}_\nu$. If we now choose $\alpha_2 = \sin\phi_1 d\phi_2 d\phi_3$ we find for the matrix of the boundary conditions

$$M_2 = \begin{pmatrix} \cos\phi_2 & 0 & \sin\phi_2 \\ 0 & 1 & 0 \\ -\sin\phi_2 & 0 & \cos\phi_2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi_1 & \sin\phi_1 \\ 0 & -\sin\phi_1 & \cos\phi_1 \end{pmatrix} \begin{pmatrix} \cos\phi_3 & -\sin\phi_3 & 0 \\ \sin\phi_3 & \cos\phi_3 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Also here we can make use of the adjoint action of the group to write $M_2 = \mathrm{Ad}(e^{\phi_2 Y_2} e^{-\phi_1 Y_1} e^{\phi_3 Y_3})$. In this case the matrix describing the boundary conditions is an element of SO(3), as obtained in section 3, and again the parameters are given by the fields themselves.

By putting M_1 and M_2 together we obtain the matrix of classical boundary conditions for the open string sigma model on $AdS_3 \times S^3$. It is important to remark that the classical boundary conditions we obtained in this way are described by a field-dependent automorphism of the corresponding Lie algebra, which preserves the metric. Thus, on the one hand, these configurations do preserve conformal invariance. On the other hand, they give rise to gluing conditions which have a similar form with the type-N gluing conditions introduced in section 3, the only difference being that, here, they are field-dependent.

We now finally come to the main point of this section, which is to investigate the possibility of having D-branes in $AdS_3 \times S^3$ which fill the whole target space. In order to see this, we must analyse the eigenvalues of the matrix of boundary conditions. We therefore need to rewrite the Neumann boundary conditions (5.3) in a slightly different form:

$$\partial X^\mu = \tilde{M}^\mu{}_\nu \bar{\partial} X^\nu \,,$$

where the field-dependent matrix $\tilde{M}$ is given by

$$\tilde{M} = \frac{G + \alpha}{G - \alpha} \,.$$

Clearly, in order to have a consistent configuration, $\tilde{M}$ should not possess -1 eigenvalues, which would correspond to Dirichlet directions. An explicit calculation shows that $\tilde{M}_1$ possesses no -1 eigenvalues, whereas $\tilde{M}_2$ does possess -1 eigenvalues, but only at the points where our parametrisation of SU(2) is singular, that is when $\cos\phi_1 = 0$. In order to explain these different results, we must take into account that, in spite of our similar treatment of the two groups SL$(2,\mathbb{R})$ and SU(2), there is however one important distinction between them: SU(2) is a compact group, whereas SL$(2,\mathbb{R})$ is noncompact. Consequently the SU(2) part of our sigma model analysis involves some subtleties, for instance our parametrisation of SU(2) in (2.2) becomes singular for $\cos\phi_1 = 0$, and the two-form B cannot be not globally defined. This indicates that, even if we impose Neumann boundary conditions in all directions the corresponding D-brane will not fill the whole S^3, but rather a three-dimensional submanifold of S^3.

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In fact, one can show that these singular points make up a disjoint union of two circles inside S^3, since

$$g_2(\pm\pi/2, \phi_2, \phi_3) = e^{\pm\pi/2 Y_1} e^{(\phi_3 \mp \phi_2) Y_3} ,$$

and hence the corresponding D3-brane in S^3 is given by the complement of the above circles in S^3. Although a direct comparison is not easy, this agrees at least morally with [24]. Hence we must conclude that one can construct D-brane configurations which fill the whole group manifold $\mathrm{SL}(2,\mathbb{R})$, but not the $\mathrm{SU}(2)$ manifold.

6. The $N = 1$ supersymmetric extension

Let us start by introducing the $N = 1$ supersymmetric extension of the affine Lie algebra $\widehat{\mathfrak{g}}$, which we will denote by

$$\widehat{\mathfrak{g}}_{N=1} = \widehat{\mathfrak{sl}}(2,\mathbb{R})_{N=1} \oplus \widehat{\mathfrak{su}}(2)_{N=1} ,$$

with generators (I_a, ψ_a) for the $\widehat{\mathfrak{sl}}(2,\mathbb{R})_{N=1}$ piece satisfying

$$I_a(z) I_b(w) = \frac{k(\eta_1)_{ab}}{(z-w)^2} + \frac{f_{ab}{}^c I_c(w)}{z-w} + \mathrm{reg} , \tag{6.1}$$

$$I_a(z) \psi_b(w) = \frac{f_{ab}{}^c \psi_c(w)}{z-w} + \mathrm{reg} , \tag{6.2}$$

$$\psi_a(z) \psi_b(w) = \frac{k(\eta_1)_{ab}}{z-w} + \mathrm{reg} . \tag{6.3}$$

and (J_a, χ_a) for the $\widehat{\mathfrak{su}}(2)_{N=1}$ piece satisfying

$$J_a(z) J_b(w) = \frac{k(\eta_2)_{ab}}{(z-w)^2} + \frac{f_{ab}{}^c J_c(w)}{z-w} + \mathrm{reg} , \tag{6.4}$$

$$J_a(z) \chi_b(w) = \frac{f_{ab}{}^c \chi_c(w)}{z-w} + \mathrm{reg} , \tag{6.5}$$

$$\chi_a(z) \chi_b(w) = \frac{k(\eta_2)_{ab}}{z-w} + \mathrm{reg} . \tag{6.6}$$

Apart from this, we also have the contribution of the free fields (φ_i, λ_i) on T^4, with the standard OPEs

$$\partial\varphi_i(z) \partial\varphi_j(w) = \frac{\delta_{ij}}{(z-w)^2} + \mathrm{reg} , \tag{6.7}$$

$$\lambda_i(z) \lambda_j(w) = \frac{\delta_{ij}}{z-w} + \mathrm{reg} . \tag{6.8}$$

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Then the generators of the $N = 1$ SCA will be given by

$$
\begin{aligned}
\mathsf{T}(z) &= \frac{1}{2k}\eta_1^{ab}(\tilde{I}_a\tilde{I}_b) + \frac{1}{2k}\eta_2^{ab}(\tilde{J}_a\tilde{J}_b) + \frac{1}{2}\sum_{i=1}^{4}(\partial\varphi_i\partial\varphi_i) + \\
&\quad + \frac{1}{2k}\eta_1^{ab}(\partial\psi_a\psi_b) + \frac{1}{2k}\eta_2^{ab}(\partial\chi_a\chi_b) + \frac{1}{2}\sum_{i=1}^{4}(\partial\lambda_i\lambda_i)\,, \\
\mathsf{G}(z) &= \frac{1}{k}\eta_1^{ab}(\tilde{I}_a\psi_b) + \frac{1}{k}\eta_2^{ab}(\tilde{J}_a\chi_b) + \sum_{i=1}^{4}(\partial\varphi_i\lambda_i) \\
&\quad - \frac{1}{6k^2}f^{abc}(\psi_a\psi_b\psi_c) - \frac{1}{6k^2}f^{abc}(\chi_a\chi_b\chi_c)\,,
\end{aligned}
$$

where

$$
\tilde{I}_a \equiv I_a - \frac{1}{2k}\eta_1^{bd}f_{ab}{}^{c}(\psi_c\psi_d) \qquad \text{and} \qquad \tilde{J}_a \equiv J_a - \frac{1}{2k}\eta_2^{bd}f_{ab}{}^{c}(\chi_c\chi_d)\,.
$$

This SCFT has a central charge $c = 15$. Notice that although so far we have only considered the holomorphic sector, we have a similar structure for the antiholomorphic sector as well. In other words, we have a $(1,1)$ SCFT.

As in the bosonic case, we consider two classes of gluing conditions. The gluing conditions of type-N are given by

$$
J_a(z) + R^b{}_a\bar{J}_b(\bar{z}) = 0\,, \qquad \psi_a(z) + S^b{}_a\bar{\psi}_b(\bar{z}) = 0\,, \tag{6.9}
$$

whereas the gluing conditions of type-D read

$$
J_a(z) - R^b{}_a\bar{J}_b(\bar{z}) = 0\,, \qquad \psi_a(z) - S^b{}_a\bar{\psi}_b(\bar{z}) = 0\,. \tag{6.10}
$$

In both cases the coefficients $R^b{}_a$ and $S^b{}_a$ are defined by $R, S : \mathfrak{g} \to \mathfrak{g}$, with $R(Z_a) = Z_bR^b{}_a$ and $S(Z_a) = Z_bS^b{}_a$, for any Z_a in $\mathfrak{g}$. These conditions are to be understood as supersymmetric generalisations of the gluing itions written down in section 3; therefore, R is taken to be an automorphism of $\mathfrak{g}$ which preserves the metric. Moreover, since we want to obtain supersymmetric configurations, the gluing conditions satisfied by the fermions will undoubtedly be related to the ones of the bosons; however we do not impose here any specific conditions on S.

These gluing conditions have to satisfy a similar consistency requirement as in the bosonic case. In this context, consistency means that the holomorphic SCFT is set equal to the antiholomorphic SCFT up to an automorphism of the $N = 1$ SCA; in other words, at the boundary we must have

$$
\mathsf{T}(z) = \bar{\mathsf{T}}(\bar{z}) \qquad \text{and} \qquad \mathsf{G}(z) = \pm\bar{\mathsf{G}}(\bar{z})\,.
$$

These boundary conditions have been written down previously in [19], in the context of Kazama-Suzuki models.

The first requirement translates into a number of conditions on the matrices R and S. Let us start with the type-D boundary conditions. From the quadratic terms in the currents we obtain that

$$R^T \eta R = \eta \,, \qquad S^T \eta R = \pm \eta \,,$$

which immediately implies that

$$S = \pm R \,, \tag{6.11}$$

as one would expect from supersymmetry. Further, from the cubic terms in the currents we have that

$$[S(Z_a), S(Z_b)] = \pm S([Z_a, Z_b]) \,, \qquad [R(Z_a), S(Z_b)] = S([Z_a, Z_b]) \,,$$

which, together with (6.11), implies that

$$[R(Z_a), R(Z_b)] = R([Z_a, Z_b]) \,. \tag{6.12}$$

In other words, the conditions that R must satisfy in order for the corresponding type-D configurations to preserve superconformal invariance match exactly the assumptions already made on R. Furthermore, it follows that these gluing conditions preserve the infinite-dimensional symmetry of the $N = 1$ current algebra (6.1)–(6.6).

If we turn now to the type-N gluing conditions, we obtain once again that $S = \pm R$, from the quadratic terms of T and G. However, from the cubic terms we obtain

$$[S(Z_a), S(Z_b)] = \mp S([Z_a, Z_b]) \,, \qquad [R(Z_a), S(Z_b)] = -S([Z_a, Z_b]) \,,$$

which, together with (6.11), implies that

$$[R(Z_a), R(Z_b)] = -R([Z_a, Z_b]) \,. \tag{6.13}$$

This implies that R must be an anti-automorphism, contrary to our Ansatz. From this we deduce that the type-N gluing conditions do *not* preserve the $N = 1$ superconformal invariance of the background. For this reason, in the remaining of this paper, we will concentrate on the type-D configurations.

From the bosonic case we know that R takes a block-diagonal form (3.6); therefore the condition (6.11) implies hat S must have a similar form

$$S = \pm \begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix} . \tag{6.14}$$

This shows that every bosonic type-D configurations that we determined previously can be made into an $N = 1$ supersymmetric configuration without having to impose additional conditions.

7. Spacetime supersymmetry

One of the most important properties of D-brane configurations is that they preserve some of the spacetime supersymmetry of the background in which they live, which translates into the fact that they satisfy the BPS condition. In the context of superconformal field theories spacetime supersymmetry appears as a by-product of $N = 2$ superconformal invariance, being related, via bosonisation, to the U(1) current. Instead of following this standard approach, here we will analyse the spacetime symmetry preserved by the D-branes we found using a different route, which was described in [13].

We will therefore consider the spacetime supercharges to be constructed directly from the $N = 1$ SCFT, by choosing five fermion bilinears and bosonising them into five scalar fields H_I, with $I = 1, \ldots, 5$ as follows

$$\partial H_1 = \frac{1}{k}(\psi_1\psi_2)\,, \qquad \partial H_2 = \frac{1}{k}(\chi_1\chi_2)\,, \qquad i\partial H_3 = \frac{1}{k}(\psi_3\chi_3)\,,$$

$$\partial H_4 = (\lambda_1\lambda_2)\,, \qquad \partial H_5 = (\lambda_3\lambda_4)\,.$$

The corresponding spacetime supercharges will then read [30]

$$Q = \oint dz e^{-\frac{\phi}{2}} S(z)\,, \tag{7.1}$$

where ϕ is the scalar field which appears in the bosonised superghost system of the fermionic string, and

$$S(z) = e^{\frac{i}{2}\sum_I \epsilon_I H_I}\,, \tag{7.2}$$

is a linear combination of the spin fields, where the coefficients $\epsilon_I = \pm 1$ label the possible supercharges, subject to a number of requirements (for a detailed discussion see [13]). Thus, due to the requirement of mutual locality between the various supercharges, and of BRST invariance these coefficients must satisfy the following conditions:

$$\epsilon_1\epsilon_2\epsilon_3 = 1\,, \qquad \epsilon_4\epsilon_5 = 1\,.$$

This yields eight supercharges for each of the holomorphic and antiholomorphic sectors of the superstring background, which are displayed in table 3.

	ϵ_1	ϵ_2	ϵ_3	ϵ_4	ϵ_5
Q_1	+	+	+	+	+
Q_2	+	-	-	+	+
Q_3	-	+	-	+	+
Q_4	-	-	+	+	+

	ϵ_1	ϵ_2	ϵ_3	ϵ_4	ϵ_5
Q_5	+	+	+	-	-
Q_6	+	-	-	-	-
Q_7	-	+	-	-	-
Q_8	-	-	+	-	-

Table 3: The spacetime supercharges of the holomorphic sector.

Given a certain D-brane configuration, we can use the gluing conditions of the fermionic fields in order to derive the boundary conditions satisfyed by the supercharges, and determine, in this way, the fraction of spacetime supersymmetry preserved by that particular boundary state. To illustrate, let us consider the case of the configurations described by a matrix R, with R_1 of the form (4.1). The corresponding conditions satisfied by the fermions read

$$\begin{aligned} \psi_1 - \cos\alpha\bar{\psi}_1 - \sin\alpha\bar{\psi}_2 &= 0\,, & \chi_1 - \cos\beta\bar{\chi}_1 - \sin\beta\bar{\chi}_2 &= 0\,,\\ \psi_2 + \sin\alpha\bar{\psi}_1 - \cos\alpha\bar{\psi}_2 &= 0\,, & \chi_2 + \sin\beta\bar{\chi}_1 - \cos\beta\bar{\chi}_2 &= 0\,,\\ \psi_3 - \bar{\psi}_3 &= 0\,, & \chi_3 - \bar{\chi}_3 &= 0\,, \end{aligned}$$

where we have systematically ignored a $\pm$ sign coming from (6.14), which does not affect the fermion bilinears in the expression of $S(z)$. Therefore we deduce that $H_I = \bar{H}_I$, for $I = 1, 2, 3$. Since we are considering D-branes embedded in $AdS_3 \times S^3$, the boundary conditions corresponding to the fermions on the torus are always the same (that is, Dirichlet), and will therefore give $H_I = \bar{H}_I$, for $I = 4, 5$. From this it follows that

$$Q_\alpha = \bar{Q}_\alpha\,, \qquad \alpha = 1, \ldots, 8 \tag{7.3}$$

This means that all the corresponding D-brane configurations discussed in section 4 preserve half of the spacetime supersymmetry of the background.

In order to analyse the spacetime supersymmetry properties of the D-brane configurations described by matrices R with R_1 of the form (4.2), we need to adopt a slightly different choice for the five fermion bilinears and, thus, for the corresponding scalar fields H_I:

$$i\partial H_1 = \frac{1}{k}(\psi_1\psi_3)\,, \qquad \partial H_2 = \frac{1}{k}(\chi_1\chi_2)\,, \qquad \partial H_3 = \frac{1}{k}(\psi_2\chi_3)\,,$$

$$\partial H_4 = (\lambda_1\lambda_2)\,, \qquad \partial H_5 = (\lambda_3\lambda_4)\,.$$

Similarly, we obtain that for the corresponding D-brane configurations the supercharges satisfy the same conditions (7.3). Due to the nonlocal nature of the dependence of the spacetime supercharges on the fermionic fields and to the particular form of the boundary conditions satisfied by the fields in the third case of the discussion in section 4 (that is, where R_1 is of the form (4.3)), it is rather difficult to determine the fraction of spacetime supersymmetry preserved this type of configurations.

To summarise, we have obtained that all the D-brane configurations which preserve the superconformal invariance of te background and have a geometrical description give rise to BPS states preserving half of the spacetime supersymmetry.

Let us conclude this section with a remark. Every boundary state that we identified and which gives rise to a Dp-brane in $AdS_3 \times S^3$ (as shown in table 2), can also describe (with appropriate boundary conditions in the 'flat' directions) D$(p+2)$- and D$(p+4)$-branes wrapped on T^2 and T^4, respectively.

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8. Discussion

In this paper we have studied, using the SCFT framework and the boundary state formalism, the possible D-brane configurations which can be consistently defined in an $AdS_3 \times S^3$ background characterised by a purely NS-NS B field. We have seen that at the bosonic level one can define two classes of gluing conditions which preserve conformal invariance, which we called type-N and type-D conditions. Type-D gluing conditions have the additional property that they preserve the infinite-dimensional symmetry of the bulk theory generated by the chiral currents.

In order to determine the geometry of the corresponding D-brane configurations we had to first obtain the boundary conditions encoded in the algebraic gluing conditions. Then, by analysing these boundary conditions, we were able to show that type-D gluing conditions describe D-branes whose worldvolume is given by shifted conjugacy classes in the group manifold. Furthermore, this type of configurations admits an $N = 1$ supersymmetric generalisation which preserves not only the superconformal invariance of the corresponding background, but also the underlying $N = 1$ affine superalgebra of the bosonic and fermionic currents. By contrast, type-N gluing conditions do not preserve the current algebra of the bulk theory. They describe D-brane configurations which are slightly more difficult to characterise geometrically, in the most general case. We have however seen that we obtain in particular D-branes whose worldvolume is an open submanifold of dimension equal to the dimension of the target manifold, and also some subgroups and cosets. This type of D-brane configurations does not seem to generalise to the supersymmetric case, in the sense that it does not yield superconformal configurations.

The two classes of bosonic D-brane configurations found here bear a certain degree of similarity with the D-brane configurations obtained in [24] using a open string WZW model analysis. There it was moreover shown that the two distinct classes of D-brane configurations are related by Poisson-Lie T-duality. It would be interesting to invest igate this possible relashionship also in our setting.

All D-brane configurations whose spacetime supersymmetry properties we have been able to analyse have in common the fact that the complex structure (implicitly defined through the choice of fermion bilinears) on the ten-dimensional background gives rise, when restricted to the tangent space of the worldvolume of a given D-brane, to a complex structure on the corresponding submanifold of the target. In other words, these D-brane configurations correspond to pseudocomplex cycles in the sense of [19].

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A. Conjugacy classes of $\mathrm{SL}(2,\mathbb{R})$ (with J.M. Figueroa-O'Farrill)

In this appendix we determine the conjugacy classes of the noncompact Lie group $\mathrm{SL}(2,\mathbb{R})$. This is probably a classic result, but we are unaware of any reference. We also analyse their causal structure relative to the natural bi-invariant metric on the group.

A.1 Jordan normal forms

We will think of $\mathrm{SL}(2,\mathbb{R})$ as the group of 2×2 real matrices with unit determinant:

$$\mathrm{SL}(2,\mathbb{R}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, ad - bc = 1 \right\} .$$

This shows that $\mathrm{SL}(2,\mathbb{R})$ is a three-dimensional Lie group, which can be represented as a hyperboloid in $\mathbb{R}^4$.

Let us embed $\mathrm{SL}(2,\mathbb{R})$ in $\mathrm{GL}(2,\mathbb{R})$, and in this way think of every element in $\mathrm{SL}(2,\mathbb{R})$ as the matrix of a linear transformation in $\mathbb{R}^2$ relative to some basis. Conjugation by $\mathrm{GL}(2,\mathbb{R})$ will then correspond to a change of basis in $\mathbb{R}^2$. The $\mathrm{GL}(2,\mathbb{R})$-orbits in $\mathrm{SL}(2,\mathbb{R})$ will then be labelled by, say, normal forms of the linear transformations. One such normal form is the *Jordan normal form*. Although this usually is presented in a way that requires a complex change of basis—that is, conjugation in $\mathrm{GL}(2,\mathbb{C})$—it is easy to restrict oneself to real changes of basis.

According to the main theorem in the theory of Jordan normal forms, any 2×2 complex matrix is conjugate under $\mathrm{GL}(2,\mathbb{C})$ to one of the following normal forms:

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \qquad \text{or} \qquad \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} .$$

If we start with a matrix which actually belongs to $\mathrm{SL}(2,\mathbb{R})$, the normal form must have unit determinant and real trace, since the trace and determinant are invariant under conjugation. That means that the normal forms of matrices in $\mathrm{SL}(2,\mathbb{R})$ are of the form

$$\begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix} \qquad \text{or} \qquad \begin{pmatrix} \pm 1 & 1 \\ 0 & \pm 1 \end{pmatrix} ,$$

where $\lambda + 1/\lambda$ is real.

If the resulting normal form is real, then it is plain to see that the matrices are actually conjugate in $\mathrm{GL}(2,\mathbb{R}) \subset \mathrm{GL}(2,\mathbb{C})$. To see this let M and M' be real 2×2 matrices which are conjugate under $\mathrm{GL}(2,\mathbb{C})$. This means that there exists some matrix $S \in \mathrm{GL}(2,\mathbb{C})$ such that

$$MS = SM' .$$

We can interpret this as a system of homogeneous linear equations for the entries of S with real coefficients. Because the constraint that the determinant of S be nonzero is an open condition, it means that we can choose the entries of S to be real. (Of course, there will be other choices of S which are complex.)

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If the resulting normal form is not real, then it is necessarily of the form

$$\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \qquad \text{with } \theta \text{ real.}$$

This matrix is itself conjugate under $\mathrm{GL}(2,\mathbb{C})$ to the real matrix

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

Hence the original matrix in $\mathrm{SL}(2,\mathbb{R})$ is conjugate to the above matrix under $\mathrm{GL}(2,\mathbb{C})$ and by the previous argument, the conjugation can actually be taken to be in $\mathrm{GL}(2,\mathbb{R})$.

In summary, the $\mathrm{GL}(2,\mathbb{R})$ orbits of $\mathrm{SL}(2,\mathbb{R})$ are labelled by the following matrices

$$\begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix}, \qquad \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \qquad \text{or} \qquad \begin{pmatrix} \pm 1 & 1 \\ 0 & \pm 1 \end{pmatrix}, \tag{A.1}$$

where we can choose θ in $[0,\pi]$ and λ real with $0 < |\lambda| \leq 1$. These choices correspond to the choice in ordering the eigenvalues of the (diagonalisable) normal forms.

In order to recover the $\mathrm{SL}(2,\mathbb{R})$ conjugacy classes from these $\mathrm{GL}(2,\mathbb{R})$ orbits it is necessary to decompose every $\mathrm{GL}(2,\mathbb{R})$ orbit into $\mathrm{SL}(2,\mathbb{R})$ orbits. Let M be a matrix in $\mathrm{SL}(2,\mathbb{R})$ and let $\mathcal{O}$ denote its $\mathrm{GL}(2,\mathbb{R})$ orbit:

$$\mathcal{O} := \left\{ gMg^{-1} \mid g \in \mathrm{GL}(2,\mathbb{R}) \right\} .$$

The Lie group $\mathrm{GL}(2,\mathbb{R})$ has two connected components corresponding to those elements with positive and negative determinant. Correspondingly $\mathcal{O}$ breaks up into two connected components: $\mathcal{O} = \mathcal{O}_+ \cup \mathcal{O}_-$, where

$$\mathcal{O}_\pm := \left\{ gMg^{-1} \mid g \in \mathrm{GL}(2,\mathbb{R}) \text{ and } \pm \det g > 0 \right\} .$$

Now every matrix $g \in \mathrm{GL}(2,\mathbb{R})$ with $\det g > 0$ can be written as

$$g = \sqrt{\det g}\, s\,, \qquad \text{where} \quad s := \frac{1}{\sqrt{\det g}}\, g\,,$$

where the matrix s has unit determinant and hence belongs to $\mathrm{SL}(2,\mathbb{R})$. Since for such a matrix g,

$$gMg^{-1} = sMs^{-1}\,,$$

we see that $\mathcal{O}_+$ is precisely the $\mathrm{SL}(2,\mathbb{R})$-orbit of M.

On the other hand, let $g \in \mathrm{GL}(2,\mathbb{R})$ with $\deg g < 0$. Then we can write it as

$$g = \sqrt{|\det g|}\, s\, t\,,$$

where t is any matrix with determinant -1, for example

$$t := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and where s is now given by

$$s := \frac{1}{\sqrt{|\det g|}}\, g\, t^{-1},$$

and has unit determinant again. Now for such a g, we have that

$$gMg^{-1} = stMt^{-1}s^{-1},$$

whence $\mathcal{O}_-$ is the $\mathrm{SL}(2,\mathbb{R})$-orbit of the matrix tMt^{-1} which belongs to $\mathrm{SL}(2,\mathbb{R})$.

In summary we see that the $\mathrm{GL}(2,\mathbb{R})$-orbit of a matrix $M \in \mathrm{SL}(2,\mathbb{R})$ breaks up in at most two $\mathrm{SL}(2,\mathbb{R})$ orbits: that of M and that of tMt^{-1}. It might happen that M and tMt^{-1} are actually in the same $\mathrm{SL}(2,\mathbb{R})$ orbit, and one has to check this case by case. At any rate, it is now a simple matter to enumerate the conjugacy classes of $\mathrm{SL}(2,\mathbb{R})$ from the enumeration of the $\mathrm{GL}(2,\mathbb{R})$ orbits in (A.1). Every matrix in $\mathrm{SL}(2,\mathbb{R})$ is conjugate in $\mathrm{SL}(2,\mathbb{R})$ to one of the following matrices:

$$\begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix} \quad \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \quad \pm\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{or} \quad \pm\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \tag{A.2}$$

where to avoid repetition we must now take λ real with $0 < |\lambda| < 1$ and $\theta \in [0, 2\pi)$.

The results are summarised in table 4. There are two conjugacy classes $\mathcal{C}_{\pm e}$ consisting of a point each, corresponding to the elements $\pm e$ in the centre of $\mathrm{SL}(2,\mathbb{R})$. The remaining conjugacy classes are two-dimensional: four one-parameter families corresponding to $\mathcal{C}_\lambda$ for $\lambda \in (0,1)$ and $\lambda \in (-1,0)$ and to $\mathcal{C}_\theta$ for $\theta \in (0,\pi)$ and $\theta \in (-\pi, 0)$, and four isolated classes $\mathcal{C}_{\pm\pm}$.

A.2 The geometry of the conjugacy classes

It is possible to understand the geometry of these conjugacy classes in $\mathrm{SL}(2,\mathbb{R})$. To this end we let us reconsider the embedding of $\mathrm{SL}(2,\mathbb{R})$ as a hyperboloid in $\mathrm{Mat}(2,\mathbb{R}) \cong \mathbb{R}^4$, this time using a different coordinate system for $\mathbb{R}^4$:

$$\mathrm{SL}(2,\mathbb{R}) = \left\{ \begin{pmatrix} x+u & y+v \\ y-v & x-u \end{pmatrix} \middle|\; x^2 + y^2 = 1 + u^2 + v^2 \right\}.$$

This embedding has the virtue of exhibiting the $\mathrm{SO}(2,2)$ isometry group of $\mathrm{SL}(2,\mathbb{R})$ manifestly. The isometries are nothing but the product of left and right transformations (modulo the centre which acts trivially):

$$\mathrm{SO}(2,2) \cong \frac{\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})}{\mathbb{Z}_2}.$$

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Class	Element	Topology	Causal type
$\mathcal{C}_e$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	point	
$\mathcal{C}_{-e}$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	point	
$\mathcal{C}_{++}$	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	$\mathbb{R} \times S^1$	degenerate
$\mathcal{C}_{--}$	$\begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}$	$\mathbb{R} \times S^1$	degenerate
$\mathcal{C}_{+-}$	$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$	$\mathbb{R} \times S^1$	degenerate
$\mathcal{C}_{-+}$	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$	$\mathbb{R} \times S^1$	degenerate
$\mathcal{C}_\theta$	$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$	$\mathbb{R}^2$	euclidean
$\mathcal{C}_\lambda$	$\begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix}$	$\mathbb{R} \times S^1$	minkowskian

Table 4: $\mathrm{SL}(2,\mathbb{R})$ conjugacy classes, with typical element, topology and causal type.

For our present purpose, a more interesting property of this embedding is that the coordinate x equals half the trace, which is an invariant of the conjugacy class. Therefore the conjugacy classes will be contained in the intersection of the three-dimensional hyperboloid

$$H_3 \quad : \quad x^2 + y^2 = 1 + u^2 + v^2 \tag{A.3}$$

with the affine hyperplanes $x =$ constant. We can distinguish several regions of interest: $|x| < 1$, $|x| = 1$, and $|x| > 1$. We will now analyse each of them in turn. The results are illustrated in figure 1.

We start with the regions $|x| > 1$. The intersection of the affine hyperplane $x =$ constant with H_3 is given by $(x^2 - 1) + y^2 = u^2 + v^2$ which is a two-dimensional one-sheeted hyperboloid. It is connected and hence it is a conjugacy class: the class $\mathcal{C}_\lambda$, where $x = \lambda + 1/\lambda$.

The intersection of H_3 with the affine hyperplane $x = \pm 1$ is the light-cone $y^2 = u^2 + v^2$. This breaks up into three conjugacy classes: the apex of the cone, which consists of the class $\mathcal{C}_{\pm e}$, the upper light-cone $y > 0$ with the apex removed, which is the class $\mathcal{C}_{\pm +}$ and the lower light-cone $y < 0$ with the apex removed, which is the class $\mathcal{C}_{\pm -}$.

Finally, for $|x| < 1$, the intersection of H_3 with the affine hyperplane $x =$ constant is the two-dimensional two-sheeted hyperboloid $y^2 = u^2 + v^2 + (1 - x^2)$. Each sheet

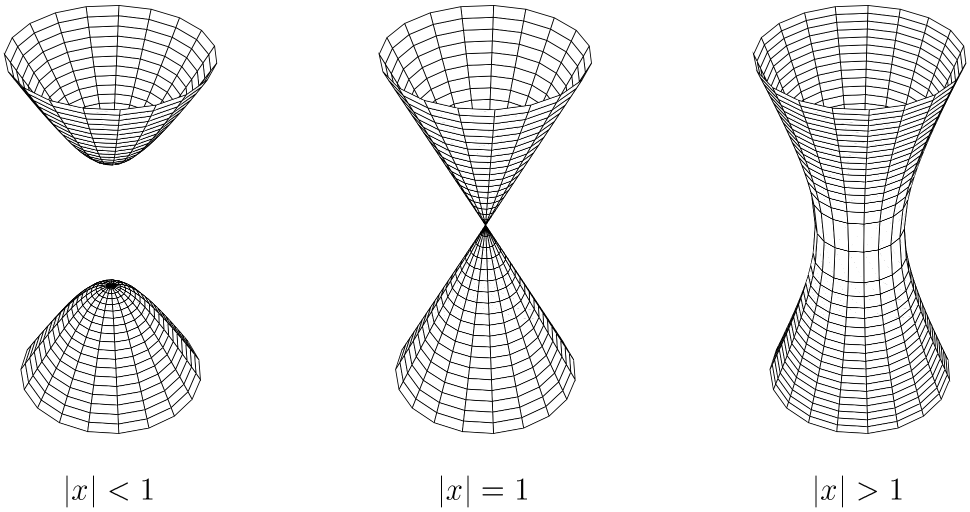

Figure 1: The different types of conjugacy classes of $\mathrm{SL}(2,\mathbb{R})$.

is one conjugacy class. If we let $x = \cos\theta$ then the upper sheet is the class $\mathcal{C}_\theta$ when $\theta \in (0, \pi)$ and the lower sheet is the class $\mathcal{C}_\theta$ when $\theta \in (-\pi, 0)$.

As a concluding comment, let us remark that the method presented here can be employed with a little extra effort to enumerate the conjugacy classes of $\mathrm{SL}(n,\mathbb{R})$.

A.3 The causal structure of the conjugacy classes

Due to their possible interpretation as D-branes, it is important to establish the causal structure of the conjugacy classes which were found above: only those classes which are nondegenerate can be straightforwardly interpreted as boundary conditions for strings. The determination of the causal structure is made easy by the fact that these classes are described by the intersection of affine hyperplanes in $\mathbb{R}^4$ with the hyperboloid H_3 in (A.3) defining the embedding of $\mathrm{SL}(2,\mathbb{R})$ in $\mathbb{R}^4$. As was mentioned above, this embedding is isometric provided we endow $\mathbb{R}^4$ with a split metric of signature $(2, 2)$. In the coordinates (x, y, u, v) chosen above, such a metric is given by

$$ds^2 = du^2 + dv^2 - dx^2 - dy^2\,. \tag{A.4}$$

It is then a simple matter to work out the induced metric on the conjugacy classes. Let us now summarise the results. Of course, we only need concern ourselves with those conjugacy classes which are not pointlike.

A.3.1 $\mathcal{C}_{\pm\pm}$

These conjugacy classes are the deleted halves of the light-cones at $x = \pm 1$. They are defined by this equation together with $y^2 = u^2 + v^2$. Let us parametrise the conjugacy class by (ϱ, ϑ) in the following way:

$$x = \pm 1\,, \qquad y = \pm\varrho\,, \qquad u = \varrho\cos\vartheta\,, \qquad v = \varrho\sin\vartheta\,.$$

The induced metric is then given by

$$ds^2 = \varrho^2 d\vartheta^2 \,,$$

which is clearly degenerate. This means that the conjugacy classes $\mathcal{C}_{\pm\pm}$ (with signs uncorrelated) cannot be interpreted as D-branes, at least straightforwardly.

A.3.2 $\mathcal{C}_\theta$

These are two-sheeted hyperboloids obtained by intersecting the affine hyperplane defined by constant x with $|x| < 1$ and the hyperboloid H_3. We parametrise these classes by (ϱ, ϑ) in the following way

$$y = \pm\sqrt{\varrho^2 + (1 - x^2)}\,, \qquad u = \varrho\cos\vartheta\,, \qquad v = \varrho\sin\vartheta\,.$$

The induced metric is then given by

$$ds^2 = \varrho^2 d\vartheta^2 + \frac{(1 - x^2)}{\varrho^2 + (1 - x^2)}\, d\varrho^2\,,$$

which is clearly euclidean. Therefore the corresponding D-branes are euclidean D-strings.

A.3.3 $\mathcal{C}_\lambda$

These are the one-sheeted hyperboloids obtained by intersecting the affine hyperplane defined by constant x with $|x| > 1$ and the hyperboloid H_3. We parametrise these classes by (y, ϑ) in the following way

$$u = \sqrt{y^2 + (x^2 - 1)}\cos\vartheta\,, \qquad v = \sqrt{y^2 + (x^2 - 1)}\sin\vartheta\,.$$

The induced metric is then given by

$$ds^2 = \big(y^2 + (x^2 - 1)\big)\, d\vartheta^2 - \frac{(x^2 - 1)}{y^2 + (x^2 - 1)}\, dy^2\,,$$

which is clearly minkowskian. Therefore the corresponding D-branes are D-strings.

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